HEAT CONDUCTION PROBLEM FOR A DOUBLY CONNECTED PLATE

A. I. Uzdalev and E. N. Bryukhanova UDC 536.21

Formulas are obtained which characterize the temperature distribution in a doubly connected plate. The small parameter method is used in the derivation.

1. Let us consider an isotropic plate which is a doubly connected domain bounded by smooth curves in planform. The parametric form of the equation of the contour L_i is [1]:

$$x = r_i(\cos\Theta + \varepsilon_i \cos n_i\Theta),$$
$$y = r_i(\sin\Theta - \varepsilon_i \sin n_i\Theta) \quad (i=1,\ 2). \tag{1}$$

Here i = 1 corresponds to the outer contour, and i = 2 to the inner, $|\varepsilon_i| < 1$.

A constant temperature M_1 is maintained on the outer side surface, and M_2 on the inner surface. The bases of the plate are heat insulated. Let us assume that the thermal characteristics of the material are independent of the temperature.

Let us establish the temperature distribution law in the plate.

The heat conductivity differential equation and the boundary conditions for the temperature function T are [2]

$$\nabla^2 T = 0, \tag{2}$$

$$T = M_i \text{ on } L_i \quad (i = 1,\ 2); \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \Theta^2}.$$

To solve the problem it is expedient to represent the condition (3) in a Cartesian x, y coordinate system:

$$T(x_i + \varepsilon_i h_i,\ y_i + \varepsilon_i k_i) = M_i \quad \text{on} \quad L_i. \tag{4}$$

The following notation is used here:

$$x_i = r_i \cos\Theta, \quad y_i = r_i \sin\Theta, \quad h_i = r_i \cos n_i\Theta, \quad k_i = -r_i \sin n_i\Theta. \tag{5}$$

TABLE 1. Temperature Values T/M at Points of a Triangular Plate ($n_1 = 2$, $\varepsilon = 1/5$)

Approximation	Θ=0°: 1,5r_2	Θ=0°: 2,1r_2	Θ=0°: 3r_2	Θ=0°: on L_1	Θ=60°: 1,2r_2	Θ=60°: 1,5r_2	Θ=60°: 2,1r_2	Θ=60°: on L_1
0	0,368	0,675	1	1,166	0,163	0,368	0,675	0,796
1	0,348	0,613	0,774	0,850	0,171	0,388	0,737	0,889
2	0,371	0,689	0,902	1,033	0,189	0,411	0,813	0,981
3	0,369	0,682	0,880	1,012	0,190	0,413	0,820	0,992

Saratov Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 1, pp. 79-83, July, 1970. Original article submitted June 30, 1969.

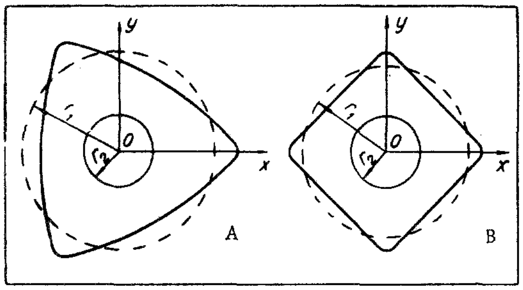

Fig. 1. View of the plates whose outer contour is determined by (1) with the parameters $r_i = r_1$, $\varepsilon_i = 1/5$, $n_i = 2$(A) and $r_i = r_1$, $\varepsilon_i = 1/9$, $n_i = 3$(B).

Let us note that x_i and y_i are coordinates of a point on a circle of radius r_i; $\varepsilon_i h_i$ and $\varepsilon_i k_i$ are magnitudes of increments in the coordinates x_i and y_i which set up the correspondence between points of the circle of radius r_i and points of the contour L_i.

Let us seek the solution of the heat conduction problem by the method of the small parameter. The quantity $\varepsilon = \varepsilon_1$ is taken as the parameter. Let us represent the desired function T as a power series in ε:

$$T=\sum_{k=0}^{N_1} \varepsilon^k T_k. \tag{6}$$

Each of the functions T_k (k = 0, 1, 2. . .) satisfies an equation of the form (2), i.e.,

$$\nabla^2 T_k = 0. \tag{7}$$

Let us derive the boundary conditions for the functions T_k. To do this we expand the temperature function in Taylor series at points of the outer and inner contours. Conditions (4) become

$$T(x_i,\ y_i)+\sum_{n=1}^{\infty} \frac{1}{n!}\ \varepsilon_i^n \Delta_n T(x_i,\ y_i) = M_i, \tag{8}$$

where

$$\Delta_n = \left(h_i \frac{\partial}{\partial x} + k_i \frac{\partial}{\partial y}\right)^n;\ \varepsilon_1=\varepsilon;\ \varepsilon_2 = m\varepsilon.$$

Here the symbolic form of writing the Taylor series is used [3]. Substituting formulas of the form (6) into conditions (8) and equating coefficients of identical powers of ε in the left and right sides, we obtain the following system of recursion relations

$$T_0(x_i,\ y_i) = M_i,$$

$$T_k(x_i,\ y_i) = -\sum_{n=1}^{k} \frac{\delta_i}{n!}\ \Delta_n T_{k-n}(x_i,\ y_i) \quad (i=1,\ 2). \tag{9}$$

The problem for a plate with noncircular outer and inner contours hence reduces to the successive integration of (7) for the functions T_k. These functions satisfy conditions (9) given on two circular contours of radius $r = r_1$ and $r = r_2$.

Limiting ourselves to the third approximation we obtain the law of the temperature distribution in the plate in the form

$$\begin{gathered} T=T_0+\varepsilon T_1+\varepsilon^2 T_2+\varepsilon^3 T_3, \\ T_0=\varphi_0(r),\ T_1=f(r^{\nu_1})\cos\nu_1\Theta+f(r^{\nu_2})\cos\nu_2\Theta, \\ T_2=\varphi_2(r)+f(r^{2\nu_1})\cos 2\nu_1\Theta+f(r^{2\nu_2})\cos 2\nu_2\Theta \\ +f(r^{\lambda})\cos\lambda\Theta+f(r^{\varkappa})\cos\varkappa\Theta, \\ T_3=F(r^{\nu_1})\cos\nu_1\Theta+f(r^{3\nu_1})\cos 3\nu_1\Theta+\Phi(r^{\nu_2})\cos\nu_2\Theta+f(r^{3\nu_2})\cos 3\nu_2\Theta \\ +f(r^{\beta_1})\cos\beta_1\Theta+f(r^{\beta_2})\cos\beta_2\Theta+f(r^{\omega_1})\cos\omega_1\Theta+f(r^{\omega_2})\cos\omega_2\Theta, \end{gathered} \tag{10}$$

TABLE 2. Temperature Values T/M at Points of a Square Plate ($n_1 = 3$, $\varepsilon = 1/9$)

Approximation	Θ=0°: $1{,}4r_2$	Θ=0°: $2r_2$	Θ=0°: on L_1	Θ=45°: $1{,}2r_2$	Θ=45°: $1{,}4r_2$	Θ=45°: on L_1
0	0,485	1	1,152	0,264	0,485	0,831
1	0,449	0,839	0,908	0,280	0,521	0,938
2	0,484	0,919	1,012	0,299	0,556	1,001
3	0,483	0,918	1,000	0,300	0,555	1,000

where

$$\varphi_k = A_k \ln r + B_k \quad (k=0,\ 2); \quad f(r^s) = C_s r^s + D r^{-s};$$
$$F(r^{\nu_1}) = C r^{\nu_1} + D r^{-\nu_1}; \quad \Phi(r^{\nu_2}) = E r^{\nu_2} + N r^{-\nu_2},$$
$$\nu_1 = n_1 + 1; \quad \nu_2 = n_2 + 1; \quad \lambda = n_1 - n_2; \quad \varkappa = n_1 + n_2 + 2; \tag{11}$$
$$\beta_1 = 2n_1 + n_2 + 3; \quad \beta_2 = 2n_2 + n_1 + 3;$$
$$\omega_1 = 2n_1 - n_2 + 1; \quad \omega_2 = 2n_2 - n_1 + 1.$$

The integration constants A_k, B_k, C_s, D_s, C, D, E, N are determined in conformity with (9) from the following conditions on the circular contours of radius $r = r_1$ and $r = r_2$:

$$T_0 = M_i,$$
$$T_1 = -\delta_i r_i \frac{dT_0}{dr} \cos(n_i+1)\Theta,$$
$$T_2 = -\delta_i \left\{ \frac{\delta_i r_i}{4} \left[\frac{d}{dr}\left(T_0 + r_i \frac{dT_0}{dr}\right) - \frac{d}{dr}\left(T_0 - r_i \frac{dT_0}{dr}\right) \cos 2(n_i+1)\Theta \right] + r_i \frac{\partial T_1}{\partial r} \cos(n_i+1)\Theta - \frac{\partial T_1}{\partial \Theta} \sin(n_i+1)\Theta \right\},$$
$$T_3 = -\delta_i \left\{ \frac{\delta_i^2 r_i}{8} \left[\frac{d}{dr}\left(-T_0 + r_i \frac{dT_0}{dr} + r_i^2 \frac{d^2T_0}{dr^2}\right) \cos(n_i+1)\Theta + \frac{d}{dr}\left(T_0 - r_i \frac{dT_0}{dr} + \frac{r_i^2}{3} \frac{d^2T_0}{dr^2}\right) \cos 3(n_i+1)\Theta \right] \right.$$
$$+ \frac{\delta_i}{2} \left[r_i^2 \frac{\partial^2 T_1}{\partial r^2} \cos^2(n_i+1)\Theta + \left(r_i \frac{\partial T_1}{\partial r} + \frac{\partial^2 T_1}{\partial \Theta^2}\right) \sin^2(n_i+1)\Theta + \sin 2(n_i+1)\Theta \frac{\partial}{\partial \Theta}\left(T_1 - r_i \frac{\partial T_1}{\partial r}\right) \right]$$
$$\left. + r_i \frac{\partial T_2}{\partial r} \cos(n_i+1)\Theta - \frac{\partial T_2}{\partial \Theta} \sin(n_i+1)\Theta \right\} \quad (i=1,\ 2), \tag{12}$$

where

$$\delta_i = \begin{cases} 1 & \text{for } i = 1, \\ m & \text{for } i = 2. \end{cases}$$

2. If the inner contour is circular, i.e., $\delta_2 = m = 0$, then (10)-(12) simplify. The temperature distribution law in the plate becomes

$$T = \varphi_0(r) + \varepsilon f(r^{\nu_1}) \cos \nu_1\Theta + \varepsilon^2 \left[\varphi_2(r) + f(r^{2\nu_1}) \cos 2\nu_1\Theta\right] + \varepsilon^3 \left[F(r^{\nu_1}) \cos \nu_1\Theta + f(r^{3\nu_1}) \cos 3\nu_1\Theta\right].$$

The coefficients in the functions φ_0, φ_2, f, F are expressed as follows:

$$A_0 = \frac{M_1 - M_2}{\ln r_1 - \ln r_2}, \quad B_0 = \frac{M_2 \ln r_1 - M_1 \ln r_2}{\ln r_1 - \ln r_2},$$

$$A_2 = \frac{H r_1^{n_1+1}}{\ln r_1 - \ln r_2}, \quad B_2 = -\frac{H r_1^{n_1+1} \ln r_2}{\ln r_1 - \ln r_2},$$

$$C_{k\nu_1} = \frac{P^{(k)} r_1^{k(n_1+1)}}{r_1^{2k(n_1+1)} - r_2^{2k(n_1+1)}}, \quad D_{k\nu_1} = -\frac{P^{(k)} r_1^{k(n_1+1)} r_2^{2k(n_1+1)}}{r_1^{2k(n_1+1)} - r_2^{2k(n_1+1)}},$$

$$C = \frac{Q r_1^{n_1+1}}{r_1^{2(n_1+1)} - r_2^{2(n_1+1)}}, \quad D = -\frac{Q r_1^{n_1+1} r_2^{2(n_1+1)}}{r_1^{2(n_1+1)} - r_2^{2(n_1+1)}}$$

$$(k = 1,\ 2,\ 3),$$

where

$$H = -(n_1+1) C_{1\nu_1}; \quad P^{(1)} = -A_0; \quad P^{(2)} = (n_1+1) r_1^{-(n_1+1)} D_{1\nu_1} + \frac{1}{2} A_0;$$

$$P^{(3)} = 2(n_1+1) r_1^{-2(n_1+1)} D_{2\nu_1} - \frac{1}{2}(n_1+1)(n_1+2) r_1^{-(n_1+1)} D_{1\nu_1} - \frac{1}{3} A_0;$$

$$Q = -\left[A_2 + 2(n_1+1) r_1^{2(n_1+1)} C_{2\nu_1} + \frac{1}{2} n_1(n_1+1) r_1^{n_1+1} C_{1\nu_1} \right].$$

Let us present the results of computing the temperature in a triangular ($n_1 = 2$, $\varepsilon = 1/5$) and a square ($n_1 = 3$, $\varepsilon = 1/9$) plate with rounded-off corners. Plates with the mentioned parameters are shown in Fig. 1. Let us consider the temperature to be T = M on the outer contour and T = 0 on the inner contour. Values of the temperature at individual points of the plates under consideration are presented in Tables 1 and 2 to the accuracy of M. The ratio r_2/r_1 for the triangular plate is taken as 1/3 and for the square as 1/2.

The accuracy of the solution obtained can be estimated by means of the error admitted in satisfying the boundary conditions. Computations showed that the greatest error is 1.2% for a triangular plate with a circular hole, and is less than 1% for the square plate. The temperature value at the outer contour is taken as 100%.

NOTATION

L_i	plate contour;
i	contour number;
x, y	coordinates;
r_i, n_i	parameters characterizing the contour size and shape;
Θ	parameter determining the position of the point on a curvilinear contour, or the polar angle on the circular contour;
ε_i	parameter characterizing the degree of deviation of the contour L_i from a circle;
M_1, M_2	temperature values on the outer and inner contours, respectively;
T	temperature function;
h_i, k_i	functions of the parameter Θ;
x_i, y_i	coordinates of a point on a circle of radius r_i;
T_0, T_1, T_2, T_3	functions governing the temperature distribution;
r	radius-vector of a point;
φ_0, φ_2, f, F, Φ	functions of the radius τ;
ν_1, ν_2, λ, $\varkappa$, β_1, β_2, ω_1, ω_2	parameters;
A_k, B_k, C_s, D_s, C, D, E, N	integration constants.

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